ISRAEL JOURNAL OF MATHEMATICS, Vol. 27, No. 1, 1977

ANALYTIC PROPERTIES OF RÉNYI'S INVARIANT DENSITY

BY

MATTHEW HALFANT

ABSTRACT

Under certain conditions a many-to-one transformation of the unit interval onto itself possesses a finite invariant ergodic measure equivalent to Lebesgue measure. The purpose of this paper is to investigate these conditions and to show how differentiable and analytic properties of the invariant density are inherited from the original transformation.

1. Introduction

Let P be a finite partition of $I = [0,1]$ specified by $0 = a_0 < a_1 < \cdots < a_p = 1$ $(p \geqq 2)$. Let T be a transformation on I which maps each element of P onto I in a monotone increasing way:

$$T(a_{i-1}+) = 0, \quad T(a_i -) = 1, \qquad i = 1, 2, \cdots, p.$$

Such a transformation is suggested in Fig. 1, drawn for $p = 3$. Notice that T^n is

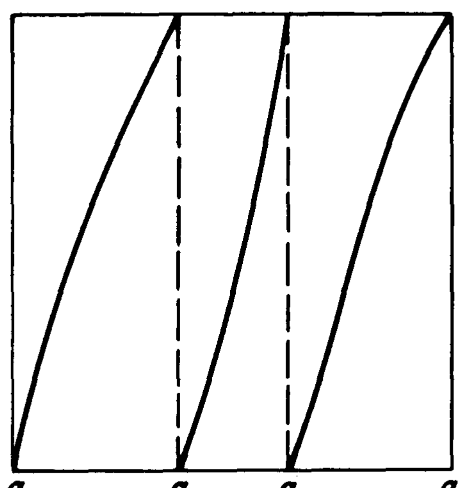

Fig. 1

Received February 26, 1976

again a transformation of the same type, there being p^n branches instead of p. In a classic paper of 1957 [5], Rényi introduced a hypothesis on T that posited the existence of an absolute constant C such that on any individual branch of T^n, for any n, the ratio of maximum to minimum slope is $\leqq C$. This was the crucial condition in Rényi's demonstration of an ergodic invariant measure ν for T:

$$\nu(T^{-1}B) = \nu(B) \quad \text{for all Borel sets } B \subset I.$$

ν is equivalent to Lebesgue measure μ, and the invariant density

$$h = \frac{d\nu}{d\mu}$$

is shown by Rényi to satisfy almost everywhere on I the inequality

$$\frac{1}{C} \leqq h \leqq C, \tag{1}$$

the constant being the same as above. We are concerned in this paper with the function h; we wish to establish its continuity, differentiability, analyticity; we show that these properties are inherited from T (or rather from T restricted to each of the elements of P). We show at the same time that h is quite generally the uniform limit of the probability density of $T^n x$ (x uniform on I) as $n \to \infty$. These results are given in section 6. In section 2 we describe the origin of this problem. Section 3 sets out the pertinent notations and precise definitions; Rényi's hypothesis, referred to above, is given in section 4. Section 5 deals with the problem of identifying the class of transformations T to which Rényi's results apply (and which therefore possess an invariant density). In section 7 we briefly indicate some natural generalizations. A final section concludes with some acknowledgements.

2. Background of the problem

Let $\phi \in C[0,1]$ be strictly increasing from $\phi(0) = 0$ to $\phi(1) = p$, an integer $\geqq 2$. Such a function can be used to associate with each $x \in [0,1)$ an infinite integer sequence $\{k_n\}$ as follows. Take for k_1 the greatest integer in $\phi(x)$, and let $r_1 = \phi(x) - k_1$ be the *remainder*. Since r_1 lies in $[0,1)$ we may form an iterative scheme:

$$k_{n+1} = [\phi(r_n)], \quad r_{n+1} = \phi(r_n) - k_{n+1}, \quad n \geqq 1.$$

Such expansions were considered in 1924 by Sôichi Kakeya [3], who in fact allowed the function ϕ to be either increasing *or* decreasing, and to have $+\infty$ as the upper limit of its range. The idea is that for

(2) $$\phi(x) = px$$

we get the ordinary expansion to base p, while for

(3) $$\phi(x) = \frac{1}{x}$$

we obtain the simple continued fraction. Kakeya wanted to unite these two kinds of expansions into a single generalization.

Of course for the generalization to be precise we must know that a number x can always be *retrieved* from its associated expansion $\{k_n\}$. Let f denote the function inverse to ϕ; then we have

$$x = f(k_1 + r_1) = f(k_1 + f(k_2 + r_2)) = \cdots = f(k_1 + f(k_2 + \cdots f(k_n + r_n) \cdots)).$$

We denote the final expression more simply by

$$x = [k_1, k_2, \cdots, k_n + r_n].$$

What is wanted is that

$$x = \lim_{n \to \infty} [k_1, k_2, \cdots, k_n].$$

To obtain a convenient formulation of this requirement, we use the notion of an *interval of rank n*, by which is meant a set of t satisfying an inequality of the form

$$[k_1, k_2, \cdots, k_n] \leqq t < [k_1, k_2, \cdots, k_n + 1].$$

The elements belonging to this interval are characterized by having k_1 through k_n as the first n terms of their expansions. I is partitioned by p^n intervals of rank n, and the longest of these, we shall say, has length Δ_n. The condition

(4) $$\lim_{n \to \infty} \Delta_n = 0$$

guarantees that two different x's won't have identical expansions. For choose n so that Δ_n is smaller than the difference between these x's; then the expansions cannot agree for the first n terms. Thus (4) states compactly that the expansions generated by ϕ are unique; Kakeya showed that this would obtain on the hypothesis that $|\phi'(x)| > 1$ for almost every $x \in I$.

Being provided now with a large class of number expansions, it becomes natural to inquire if the statistical regularity of the terms $\{k_n\}$, well known in the case of base p expansions (Borel) and continued fractions (Kuz'min [4]), persists into the broader setting drawn by Kakeya. The ergodic methods used by

Ryll-Nardzewski [6] are clearly the appropriate tool, and their decisive application was in fact made by Rényi [5] in 1957.

Beyond restrictions on ϕ designed purely to insure $\Delta_n \to 0$ ($|\phi'|$ always $\geqq 1$, but allowed to $= 1$ on special subintervals of I) Rényi was forced to impose the hypothesis already mentioned, and spelled out below in section 4. This allowed the conclusion that there exists a measurable function h, bounded away from 0 and from ∞, such that for almost every $x \in I$, the sequence of remainders $\{r_n\}$ is distributed on I with density h. The regularity of the sequences $\{k_n\}$ follows at once.

In this paper we study the transformation

$$T : x \to \phi(x) - [\phi(x)]$$

carrying each remainder into the next. Our results bear some relation to generalizations of the base p expansion (2). Our methods are readily adapted to the continued fraction (3) and its generalizations; see section 7.

3. Notation, terminology, definitions

Let p be an integer $\geqq 2$, and partition $I = [0,1]$ by the points

$$0 = a_0 < a_1 < \cdots < a_p = 1.$$

Let T map $I \to I$. Denote by T_i the restriction of T to $[a_{i-1}, a_i]$, modified at the endpoints so that

$$T_i(a_{i-1}) = 0, \quad T_i(a_i) = 1.$$

We will say that the transformation T is *admissible* if the following conditions hold:

1) $T(0) = 0$, $T(1) = 1$; for $0 < a_i < 1$, $T(a_i) =$ either 0 or 1.
2) $T_i \in C^2[a_{i-1}, a_i] \quad \forall i$.
3) $T_i' > 0$ on $[a_{i-1}, a_i] \quad \forall i$.

Condition (1) is not essential, but we adopt it to simplify the statement of certain results (see Theorem 4 and 5). Condition (2) is stronger than it need be, and likewise condition (3); see section 7. We denote the class of admissible transformations by $\mathcal{A}$.

For each i, let f_i denote the function inverse to T_i. Thus $f_i \in C^2[0,1]$, $f_i' > 0$ on I, and

$$f_i(0) = a_{i-1}, \quad f_i(1) = a_i.$$

The transformation T^n maps $I \to I$ p^n-to-1; thus the inverse mapping has p^n branches, of which we denote the ith by $f_{n,i}$. We define $f_{n,i}$ at 0 and 1 so as to make the result continuous on I.

The *intervals of rank n* are the intervals $[f_{n,i}(0), f_{n,i}(1))$, $i = 1, 2, \cdots, p^n$. The collection of all such intervals, over all $n \geqq 1$, is known as the class of *fundamental intervals.* We shall refer to the numbers $f_{n,i}(0)$, $f_{n,i}(1)$ as fundamental endpoints.

We denote by Δ_n the length of the largest interval of rank n:

$$\Delta_n = \max_i \{f_{n,i}(1) - f_{n,i}(0)\}.$$

Let μ denote Lebesgue measure on I. For $n = 0, 1, 2, \cdots$ we define the *nth iterated distribution* Φ_n by

$$\Phi_n(t) = \mu(\{x \in I : r_n = T^n x \leqq t\}) = \mu(T^{-n}[0, t]), \quad t \in I.$$

The *iterated densities* S_n are defined almost everywhere on I by

$$S_n(t) = \Phi_n'(t).$$

The *invariant measure* ν is a probability measure, whose existence is guaranteed for admissible transformations satisfying Rényi's condition (see below), such that for arbitrary measurable sets $E \subset I$,

$$\nu(T^{-1}E) = \nu(E).$$

The *invariant distribution* Φ is defined by

$$\Phi(t) = \nu([0, t]), \quad t \in I,$$

and *Rényi's invariant density* h is defined by

$$h(t) = \Phi'(t), \quad t \in I.$$

4. Rényi's condition

Put

$$\frac{\sup_{t \in I} f'_{n,i}(t)}{\inf_{t \in I} f'_{n,i}(t)} = C_{n,i} \leqq \infty$$

and

$$\max_i C_{n,i} = C_n.$$

Rényi's condition is the requirement

$$\sup_n C_n = C < \infty.$$

5. The class $\mathscr{R}$ of Rényi transformations

We denote by $\mathscr{R}$ the subclass of admissible transformations which fulfill Rényi's condition. While Rényi himself gave no indication of the nature of this class, the following result appears to be well known among present-day ergodic theorists.

THEOREM 1. *If $T \in \mathscr{A}$ and $\inf_{t\in I} T'(t) > 1$, then $T \in \mathscr{R}$.*

NOTE. T' is not defined at the points a_i; the condition above is an abbreviation for

$$\min_i \inf_{t\in[a_{i-1}, a_i]} T_i'(t) > 1.$$

PROOF. Here and in what follows we will make use of the functional relationship

$$f_{n+1,i}(t) = f_j(f_{n,k}(t)) \tag{5}$$

where i on the left determines j and k on the right, and conversely. Differentiating (5) and using the notation of section 4, we obtain in an obvious manner

$$C_{n+1,i} \leqq C_{n,k} \cdot \sup \frac{f_j'(f_{n,k}(t_1))}{f_j'(f_{n,k}(t_2))},$$

the supremum being taken over all $t_1, t_2 \in I$. We can estimate this supremum by noting that the arguments of f_j' differ by a quantity not larger in magnitude than Δ_n. Writing

$$M = \max_j \sup_{t\in I} |f_j''(t)| < \infty,$$

$$\alpha = \min_j \inf_{t\in I} f_j'(t) > 0,$$

we have

$$1 \leqq \sup \frac{f_j'(f_{n,k}(t_1))}{f_j'(f_{n,k}(t_2))} \leqq 1 + \frac{M}{\alpha}\Delta_n$$

by the mean value theorem. Thus

$$C_{n+1,i} \leqq C_{n,k}\left(1+\frac{M}{\alpha}\Delta_n\right).$$

Taking the supremum first over k on the right, then over i on the left, gives

$$C_{n+1} \leqq C_n\left(1+\frac{M}{\alpha}\Delta_n\right).$$

Note that C_1 is finite; in fact $C_1 \leqq \beta/\alpha$ where

$$\beta = \max_j \sup_{t\in I} f_j'(t).$$

Therefore

$$\sup_n C_n \leqq C_1 \prod_{n=1}^{\infty}\left(1+\frac{M}{\alpha}\Delta_n\right)$$

will be finite provided the product on the right converges; this is equivalent to the convergence of the series

$$\sum_{n=1}^{\infty} \Delta_n.$$

We establish this by showing that, for all n, $\Delta_n \leqq \beta^n$. (By our hypothesis, $\beta < 1$.) The length of a typical interval of rank 1 is

$$f_i(1)-f_i(0)=f_i'(\theta)\leqq \beta$$

by the mean value theorem. Hence $\Delta_1 \leqq \beta$. An induction can be based on (5); assuming $\Delta_l \leqq \beta^l$ for $l \leqq n$, we write the length of an interval of rank $n+1$ as

$$\begin{aligned} f_{n+1,i}(1)-f_{n+1,i}(0) &= f_j(f_{n,k}(1))-f_j(f_{n,k}(0)) \\ &= f_j'(\theta)(f_{n,k}(1)-f_{n,k}(0)) \leqq \beta\Delta_n \leqq \beta^{n+1} \end{aligned}$$

so that $\Delta_{n+1} \leqq \beta^{n+1}$, completing the proof. We have shown that Rényi's condition obtains with the constant

$$C \leqq \frac{\beta}{\alpha}\prod_{n=1}^{\infty}\left(1+\frac{M}{\alpha}\beta^n\right).$$

Consider now the question of what happens when the slope T' can assume values $\leqq 1$. Our previous argument already shows that $T \in \mathcal{R}$ if

$$\sum_{n=1}^{\infty} \Delta_n < \infty;$$

the *necessity* of this condition is more troublesome to establish. However, when we consider slopes $\leqq 1$ we move out of range of Kakeya's theorem, and we will at the very least want to assure ourselves that $\Delta_n \to 0$.

THEOREM 2. *An admissible transformation belongs to* $\mathcal{R}$ *if, and only if,*

$$\sum_{n=1}^{\infty} \Delta_n < \infty.$$

PROOF. Sufficiency has been shown; necessity is proved in two steps. First we show that $T \in \mathcal{R} \Rightarrow \Delta_n \to 0$. For, if this is not the case, then we must have $\Delta_n \to c > 0$, since the sequence $\{\Delta_n\}$ is monotonically decreasing. Thus we must be able to produce an interval $[\sigma, \tau] \subset I$ with $\tau - \sigma = c$, and in whose interior no fundamental endpoint can be found. It is possible that i) σ is such an endpoint, but not τ; ii) τ is such, but not σ; or iii) neither is. Assume (iii). Then there must be fundamental endpoints arbitrarily close on either side. Given $\varepsilon > 0$ there exists n such that $[f_{n,i}(0), f_{n,i}(1)] \supset [\sigma, \tau]$, the corresponding endpoints of these intervals being separated by less than ε (see Fig. 2). Now $f_{n,i}(\sigma) \in (f_{n,i}(0), f_{n,i}(1)) \subset [\sigma - \varepsilon, \tau + \varepsilon]$; however we cannot have $f_{n,i}(\sigma) \in (\sigma, \tau)$, for then also $f_{n,i}(f_{m,j}(0)) \in (\sigma, \tau)$ for some m, j. But $f_{n,i}(f_{m,j}(0)) = f_{n+m,k}(0)$ is a fundamental endpoint, contradicting the construction of $[\sigma, \tau]$. Thus we must have either a) $f_{n,i}(\sigma) \in [\sigma - \varepsilon, \sigma]$ or b) $f_{n,i}(\sigma) \in [\tau, \tau + \varepsilon]$. On (a),

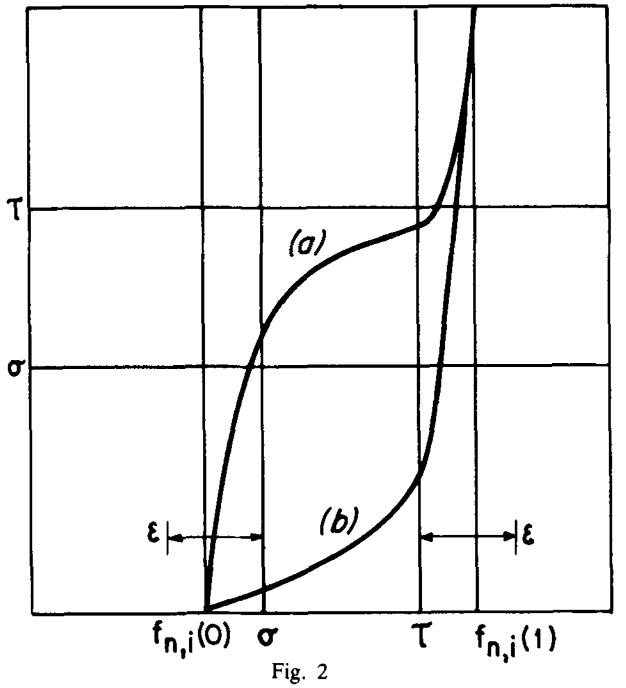

Fig. 2

$$f_{n,i}(\sigma)-f_{n,i}(0)=f'_{n,i}(\xi)\sigma\leqq\varepsilon$$

and

$$f_{n,i}(1)-f_{n,i}(\sigma)=f'_{n,i}(\eta)(1-\sigma)\geqq c,$$

so

$$\frac{f'_{n,i}(\eta)}{f'_{n,i}(\xi)}\geqq\frac{\sigma c}{1-\sigma}\cdot\frac{1}{\varepsilon};$$

because ε is arbitrary, this violates Rényi's condition. Likewise, on (b),

$$f_{n,i}(\sigma)-f_{n,i}(0)=f'_{n,i}(\xi)\sigma\geqq c$$

and

$$f_{n,i}(1)-f_{n,i}(\sigma)=f'_{n,i}(\eta)(1-\sigma)\leqq\varepsilon,$$

so

$$\frac{f'_{n,i}(\xi)}{f'_{n,i}(\eta)}\geqq\frac{c(1-\sigma)}{\sigma}\cdot\frac{1}{\varepsilon},$$

again contradicting Rényi's condition.

On cases (i) and (ii), the same kind of argument applies.

Now we establish convergence of the series $\Sigma\,\Delta_n$. We have

$$f_{n,i}(1)-f_{n,i}(0)=f'_{n,i}(\theta)$$

so that, at least for one particular $\theta\in(0,1)$,

$$f'_{n,i}(\theta)\leqq\Delta_n.$$

In view of Rényi's condition, this implies that

$$\sup_{t\in I}f'_{n,i}(t)\leqq C\Delta_n. \tag{6}$$

Again,

$$f_{n,i}(f_{n,j}(1))-f_{n,i}(f_{n,j}(0))=f'_{n,i}(\theta)(f_{n,j}(1)-f_{n,j}(0))\leqq C\Delta_n^2.$$

Taking the supremum over i and j on the left gives

$$\Delta_{2n}\leqq C\Delta_n^2.$$

Since $\Delta_n\to 0$ we can choose N such that $C\Delta_N=\rho<1$. Then $\Delta_{2N}\leqq C\Delta_N^2$, $\Delta_{4N}\leqq C\Delta_{2N}^2\leqq C(C\Delta_N^2)^2=C^3\Delta_N^4$, and, by induction,

$$\Delta_{2^nN}\leqq C^{2^n}\Delta_N^{2^n}/C=\rho^{2^n}/C.$$

Consider those terms Δ_k for which k lies between 2^nN and $2^{n+1}N$. There are essentially 2^nN such terms, all $\leqq \Delta_{2^nN}$. By the above inequality, the sum of these terms cannot exceed $2^nN\rho^{2^n}/C$. It follows that the series $\Sigma\,\Delta_n$ must converge.

While we have stated Theorem 2 in a form convenient for use, let us note that we have actually proved

THEOREM 2′. *An admissible transformation belongs to* $\mathcal{R}$ *if, and only if,* $\Delta_n = O(\vartheta^n)$ *for some* $\vartheta \in (0,1)$.

As an application of Theorem 2, we prove

THEOREM 3. *If* $\mathcal{A}_n$ *denotes the subclass of* $\mathcal{A}$ *whose elements* T *satisfy*

$$\inf_{\xi\in I}\prod_{k=1}^{n} T'(T^{k-1}\xi) > 1,$$

then $\mathcal{A}_n \subset \mathcal{R}$. *Moreover, any* $T \in \mathcal{R}$ *necessarily belongs to some* $\mathcal{A}_n$.

PROOF. Given n, we have for some i and some $\theta \in (0,1)$

$$\Delta_n = f_{n,i}(1) - f_{n,i}(0) = f'_{n,i}(\theta).$$

Now

$$\frac{1}{f'_{n,i}(\theta)} = \frac{d}{dt}\,T^n(t)\bigg|_{t=\xi}$$

where $\xi = f_{n,i}(\theta)$. Thus, for a definite $\xi \in I$ we have

$$\frac{1}{\Delta_n} = \prod_{k=1}^{n} T'(T^{k-1}\xi).$$

From this we have

$$\frac{1}{\Delta_n} \geqq \alpha\beta^{[n/N]}$$

where

$$\beta = \inf_{\xi\in I}\prod_{k=1}^{N} T'(T^{k-1}\xi)$$

and

$$\alpha = \inf_{\substack{\xi\in I\\ m<N}}\prod_{k=1}^{m} T'(T^{k-1}\xi).$$

If $T \in \mathcal{A}_N$ then $\beta > 1$; therefore $\Sigma\,\Delta_n < \infty$ and $T \in \mathcal{R}$.

Conversely, if $T \in \mathscr{R}$, then (6) and the fact that $\Delta_n \to 0$ assure the existence of N such that

$$\sup_{\theta \in I} f'_{N,i}(\theta) < 1$$

for each i; that is to say, $T \in \mathscr{A}_N$.

We may paraphrase this result as follows.

THEOREM 3'. *An admissible transformation T belongs to $\mathscr{R}$ if, and only if, some iterate of T has derivative everywhere* > 1.

NOTE. We are being informal here; see the remark following the statement of Theorem 1.

By way of illustration, consider the transformation

$$T: x \to 2x + \frac{3}{10}\sin(2\pi x) \pmod 1 \tag{7}$$

which has a minimum slope of $2 - 3\pi/5 \approx .115$; we find by numerical methods that $T \in \mathscr{A}_5$ and hence that $T \in \mathscr{R}$.

Theorem 3' shows as well that certain transformations do *not* belong to $\mathscr{R}$. Thus $T: x \to x + x^2 \pmod 1$ is not in $\mathscr{R}$, for any iterate of T will have derivative $= 1$ at $x = 0$. More generally we have

THEOREM 4. *Let $T \in \mathscr{A}$, and let $V = \{x \in I : T_i'(x) \leqq 1$ for some $i\}$. Then T cannot belong to $\mathscr{R}$ if V contains an orbit under T.*

PROOF. Suppose $T^m\xi \in V$ for $m = 0, 1, 2, \cdots$. Then

$$\frac{d}{dt} T^n(t)\Big|_{t=\xi} = \prod_{k=1}^{n} T'(T^{k-1}\xi) \leqq 1;$$

the conclusion follows from Theorem 3'.

We see from the above that an admissible transformation will certainly belong to $\mathscr{R}$ if its slope is everywhere > 1, and may or may not belong to $\mathscr{R}$ if smaller slopes occur. The following result brings out the delicacy of the question in the borderline case where a minimum slope of 1 occurs at a finite number of points.

THEOREM 5. *Let $T \in \mathscr{A}$ be such that* $\inf T'(t) = 1$, *and suppose that the set $U = \{x \in I : T_i'(x) = 1$ for some $i\}$ consists of finitely many points. Then $T \in \mathscr{R}$ if, and only if, U contains no orbit under T.*

Despite this "delicacy", we have the following result, which is easily shown by constructing $\mathscr{A}_2$ transformations similar in appearance to (7).

THEOREM 6. *There exist transformations in $\mathcal{R}$ whose derivative remains arbitrarily small on an interval of length arbitrarily close to* 1.

For proofs of these results, and for further discussion, we refer the reader to our dissertation [2].

6. Major results

The following theorem imitates the result of Kuz'min [4]. Our proof is modelled directly on that given by Billingsley [1] in connection with the continued fraction†.

THEOREM 7. *Let $T \in \mathcal{R}$. Then the iterated distributions Φ_n converge uniformly, as $n \to \infty$, to the invariant distribution Φ.*

PROOF. We begin by showing that the transformation T is *mixing*, which means that for arbitrary Borel sets A, B in I,

$$\lim_{n \to \infty} \nu(T^{-n}A \cap B) = \nu(A)\nu(B).$$

(Recall that ν is the invariant measure; Lebesgue measure is denoted by μ.) Let $\mathcal{G}_n$ be the σ-algebra consisting of sets of the form $T^{-n}A$, A being a Borel set in I. The $\mathcal{G}_n$ form a decreasing sequence of σ-algebras; the limit

$$\mathcal{G} = \bigcap_{n=1}^{\infty} \mathcal{G}_n$$

is called the tail σ-algebra. It can be shown (see [1, p. 121]) that T is mixing if $\mathcal{G}$ contains only sets of measure 0 or 1. We verify now that this is the case.

Let $[a, b]$ be a subinterval of I, and let D_n denote the interval

$$[f_{n,i}(0), f_{n,i}(1))$$

of rank n. Using the customary notation for conditional probability, we have

$$\mu(T^{-n}[a, b] \mid D_n) = \frac{f_{n,i}(b) - f_{n,i}(a)}{f_{n,i}(1) - f_{n,i}(0)}$$

$$= \frac{f'_{n,i}(\theta_1)(b-a)}{f'_{n,i}(\theta_2)}$$

where $\theta_1 \in (a, b)$ and $\theta_2 \in (0, 1)$. By Rényi's condition,

†This result also occurs in *F-expansions revisited* by R. L. Adler which appears in Recent Advances in Topological Dynamics, Lect. Notes in Math. **318** (1973).

$$\frac{b-a}{C} \leqq \mu(T^{-n}[a,b] \mid D_n) \leqq C(b-a).$$

It is clear that we may replace $[a,b]$ by an arbitrary Borel set $A \subset I$:

$$\frac{\mu(A)}{C} \leqq \mu(T^{-n}A \mid D_n) \leqq C\mu(A).$$

Using (1) and the fact that $h = d\nu/d\mu$, we may convert this to an inequality involving ν; this works out to be

$$\frac{\nu(A)}{C^4} \leqq \nu(T^{-n}A \mid D_n) \leqq C^4\nu(A). \tag{8}$$

Suppose now that A is a set in the tail σ-algebra $\mathcal{G}$. Then for any n, there is a Borel set B such that $A = T^{-n}B$. Thus from (8) we obtain

$$\frac{\nu(A)}{C^4} = \frac{\nu(T^{-n}B)}{C^4} = \frac{\nu(B)}{C^4} \leqq \nu(T^{-n}B \mid D_n) = \nu(A \mid D_n).$$

If $\nu(A) > 0$, then we may write

$$\nu(D_n) = \frac{\nu(A)\nu(D_n \mid A)}{\nu(A \mid D_n)} \leqq C^4\nu(D_n \mid A). \tag{9}$$

The fundamental intervals may be used to generate the Borel sets; therefore from (9) we deduce

$$\nu(E) \leqq C^4\nu(E \mid A)$$

for an arbitrary Borel set E. Taking E to be the complement of A, we find that $\nu(E) = 0$, which is to say that $\nu(A) = 1$. This shows that $\mathcal{G}$ contains only sets of measure 0 or 1, and hence that the transformation T is mixing.

We now demonstrate that, for arbitrary measurable $A \subset I$,

$$\lim_{n\to\infty} \mu(T^{-n}A) = \nu(A), \tag{10}$$

and to do this we use the equality

$$\mu(T^{-n}A) = \int_{T^{-n}A} d\mu = \int_{T^{-n}A} \frac{1}{h}\, d\nu.$$

The fact that T is mixing can be written

$$\nu(T^{-n}A \cap B) = \int_{T^{-n}A} \chi_B d\nu \to \nu(A)\nu(B) \quad \text{as} \quad n \to \infty,$$

χ_B being the characteristic function of the set B. By taking linear combinations of such characteristic functions, we conclude that

$$\lim_{n\to\infty}\int_{T^{-n}A} s\,d\nu = \nu(A)\int_I s\,d\nu \tag{11}$$

for an arbitrary simple function s. Since $1/h$ is measurable and bounded, we may approximate it *uniformly* by simple functions. Therefore we say replace s in (11) by $1/h$ and write

$$\lim_{n\to\infty}\mu(T^{-n}A) = \lim_{n\to\infty}\int_{T^{-n}A}\frac{1}{h}\,d\nu = \nu(A)\int_I \frac{1}{h}\,d\nu = \nu(A),$$

proving (10). Taking, for A, the interval $[0,t]$, we conclude

$$\lim_{n\to\infty}\Phi_n(t) = \nu([0,t]) = \Phi(t).$$

That the convergence is uniform follows automatically from the fact that all the Φ_n, and Φ as well, are both continuous and monotone. This completes the proof.

Using this result, we begin our assault on the invariant density.

THEOREM 8. *Let $T\in\mathcal{R}$. Then h is continuous on I, and is the uniform limit of the iterated densities.*

PROOF. We begin by showing that the iterated densities conform to the same inequality (1) that bounds the invariant density h. Applying Rényi's condition to the formula

$$S_n(t) = \sum_{i=1}^{p^n} f'_{n,i}(t)$$

we obtain

$$\sup_{t\in I} S_n(t) \leqq C \inf_{t\in I} S_n(t).$$

But since $\int_0^1 S_n(t)dt = 1$, we must have $\inf S_n(t)\leqq 1$, so that

$$S_n(t)\leqq C \quad \text{for} \quad t\in I.$$

Likewise we find that $S_n(t)\geqq 1/C$ on I.

Next we show that there necessarily exists an N such that

$$\sup_{t\in I}\sum_{i=1}^{p^N}(f'_{N,i}(t))^2 < 1. \tag{12}$$

We have, certainly

$$\sum_i (f'_{N,i}(t))^2 \leqq \left(\sup_{t\in I} f'_{N,i}(t)\right)\sum_i f'_{N,i}(t).$$

The first term on the right is $\leqq C\Delta_N$ by (6); the second term is just $S_N(t)$ which is $\leqq C$ as we have seen. Therefore the left hand side of (12) is $\leqq C^2\Delta_N$ which, by Theorem 2′, can be rendered arbitrarily small by choosing N sufficiently large. In what follows, we suppose N to be such that (12) obtains.

The following identity is true for any nonnegative integer k:

$$S_{k+N}(t) = \sum_{i=1}^{p^N} S_k(f_{N,i}(t)) f'_{N,i}(t).$$

By differentiation we obtain

$$S'_{k+N}(t) = \sum_i \{S'_k(f_{N,i}(t))(f'_{N,i}(t))^2 + S_k(f_{N,i}(t)) f''_{N,i}(t)\}. \tag{13}$$

Put

$$d = \sup_{t \in I} \sum_i |f''_{N,i}(t)| < \infty$$

and for $n = 0, 1, 2, \cdots$

$$B_n = \sup_{t \in I} |S'_n(t)|.$$

Let $\theta < 1$ denote the value of the left hand side of (12). From (13) we get

$$B_{k+N} \leqq B_k\theta + Cd,$$

and from this we infer the sequence

$$B_k, \quad B_{k+N}, \quad B_{k+2N}, \cdots$$

is uniformly bounded by some number $\hat{B}_k$. We conclude that the entire sequence $\{B_n\}$, $n = 0, 1, 2, \cdots$ is bounded by $B = \max\{\hat{B}_0, \hat{B}_1, \cdots, \hat{B}_{N-1}\}$.

Thus we have now established that the sequence $\{S_n\}$ of iterated densities is uniformly bounded and equicontinuous. By the Ascoli theorem there must exist a subsequence $\{\mathcal{S}_n\}$ with continuous limit:

$$\lim_{n \to \infty} \mathcal{S}_n(t) = g(t), \text{ say.}$$

Since the convergence is uniform on I, we may integrate to obtain

$$\lim_{n \to \infty} \int_0^t \mathcal{S}_n(\tau)\, d\tau = G(t), \qquad t \in I.$$

But $\{\int_0^t \mathcal{S}_n(\tau)\, d\tau\}$ is a subsequence of $\{\Phi_n(t)\}$, and so from Theorem 7 we conclude that $G = \Phi$, whence $G' = g$ must be equal to the invariant density h. Thus h is continuous on I.

To show that the full sequence of iterated densities converges uniformly to h, assume the contrary. Then it must be possible to find an $\varepsilon > 0$ and a subsequence $\{\mathcal{S}_n\}$ of $\{S_n\}$ such that

$$\sup_{t \in I} |\mathcal{S}_n(t) - h(t)| \geqq \varepsilon \qquad \text{for all } n. \tag{14}$$

The sequence $\{\mathcal{S}_n\}$ is itself uniformly bounded and equicontinuous, and so must possess a uniformly convergent subsequence with limit, say g. As before, we find that $g = h$, and this is incompatible with (14).

By a simple extension of the same reasoning, we see that h will be "nearly" as differentiable as T.

THEOREM 9. *Let $T \in \mathcal{R}$ and suppose that each $f_i \in C^m[0,1]$, $2 \leqq m \leqq \infty$. Then $h \in C^{m-2}[0,1]$. Furthermore, for $k \leqq m-2$, $S_n^{(k)}$ converges uniformly to $h^{(k)}$ as $n \to \infty$.*

PROOF. Assume $m \geqq 3$ so that we may take the derivative on both sides of (13):

$$S''_{k+N} = \sum_i \{S''_k \cdot (f_{N,i})^3 + 3S'_k \cdot (f'_{N,i} f''_{N,i}) + S_k \cdot f'''_{N,i}\}. \tag{15}$$

Put

$$\alpha = \sup_{t \in I} \sum_i (f'_{N,i}(t))^3 < \theta < 1,$$

$$\beta = \sup_{t \in I} \sum_i 3|f'_{N,i}(t) f''_{N,i}(t)| < \infty,$$

and

$$\gamma = \sup_{t \in I} \sum_i |f'''_{N,i}(t)| < \infty.$$

We know the S_n to be uniformly bounded by C, and the $|S'_n|$ to be uniformly bounded by B. If we put, for $n = 0, 1, 2, \cdots$

$$A_n = \sup_{t \in I} |S''_n(t)|,$$

then we read from (15) that

$$A_{k+N} \leqq A_k \alpha + B\beta + C\gamma.$$

The last two terms being constants, and $\alpha < \theta$ being smaller than 1, we perceive as before a uniform bound $\hat{A}_k$ to the sequence

$$A_k, \quad A_{k+N}, \quad A_{k+2N}, \cdots .$$

Also as before, we produce an absolute bound A for the entire sequence $\{A_n\}$, $n = 0, 1, 2, \cdots$.

Thus the sequence $\{S'_n\}$ of first derivatives is uniformly bounded and equicontinuous, and accordingly possesses a uniformly convergent subsequence $\{\mathcal{S}'_n\}$:

$$\lim_{n\to\infty} \mathcal{S}'_n(t) = g'(t), \text{ say.}$$

Integrating twice and using the same argument as before, we find that $g = h$ so that $h' = g'$ exists and is continuous. Again as before, we can show that the complete sequence $\{S'_n\}$ converges uniformly to h' as $n \to \infty$.

It is apparent that an induction can be made to proceed with further differentiations of (15); we may therefore regard the proof as complete.

Our final result shows that analytic Rényi transformations give rise to analytic invariant densities.

THEOREM 10. *Let $T \in \mathcal{R}$, and suppose that f_i is analytic on I for each i. Then h is analytic on I. Moreover, for $k \geqq 0$, $h^{(k)}$ is the uniform limit of $S_n^{(k)}$ as $n \to \infty$.*

PROOF. Since the uniform convergence of the derivatives is covered already by Theorem 9, only the analyticity of h needs to be shown. This we shall demonstrate first under the assumption of an additional hypothesis, namely that for each i,

$$\sup_{t\in I} f'_i(t) < 1. \tag{16}$$

For a given $\varepsilon > 0$, let $\mathcal{D} = \mathcal{D}(\varepsilon)$ denote the region of the complex plane all of whose boundary points are at distance ε from I:

$$\mathcal{D} = \{z \in \mathbf{C}\colon z = t + \varepsilon\zeta, t \in I, |\zeta| < 1\}.$$

It is clearly possible to choose ε sufficiently small that

$$\max_i \sup_{t\in\mathcal{D}} |f'_i(t)| = \beta < 1, \tag{17}$$

and we do so. Our plan is to show that the iterated densities are analytic and uniformly bounded in $\mathcal{D}$; since convergence is known to occur on $I \subset \mathcal{D}$, the analyticity of the limit h follows from the Vitali convergence theorem (see Titchmarsh [7], p. 168).

The significance of (17) is that each f_i *contracts* the set $\mathcal{D}$ under mapping; that is,

$$t \in \mathcal{D} \Rightarrow f_i(t) \in \mathcal{D}.$$

This implies through an inductive argument that all the S_n are analytic in $\mathcal{D}$; for $S_0(t) \equiv 1$, and

$$S_{n+1}(t) = \sum_i S_n(f_i(t))f_i'(t)$$

for $n \geqq 0$. The same argument applied to (5) establishes the analyticity of $f_{n,i}$, $i = 1, 2, \cdots, p^n$; $n = 1, 2, 3, \cdots$. Thus it is meaningful to define, for each natural number n,

$$\hat{\Delta}_n = \max_i \sup_{t_1, t_2 \in \mathscr{D}} |f_{n,i}(t_1) - f_{n,i}(t_2)|.$$

A simple inductive argument shows that

$$\hat{\Delta}_n \leqq K\beta^n, \qquad n = 1, 2, 3, \cdots, \tag{18}$$

where $K = \operatorname{diam}(\mathscr{D}) = 1 + 2\varepsilon$.

Using (18) it is easy to recast the proof of Theorem 1 so as to establish "Rényi's condition" in $\mathscr{D}$:

$$\frac{\sup_{t \in \mathscr{D}} |f'_{n,i}(t)|}{\inf_{t \in \mathscr{D}} |f'_{n,i}(t)|} = \hat{C} < \infty.$$

This same constant $\hat{C}$ is the required uniform bound for the iterated densities on $\mathscr{D}$:

$$\sup_{t \in \mathscr{D}} |S_n(t)| = \sup_{t \in \mathscr{D}} \left| \sum_i f'_{n,i}(t) \right|$$

$$\leqq \sup_{t \in \mathscr{D}} \sum_i |f'_{n,i}(t)| \leqq \sum_i \sup_{t \in \mathscr{D}} |f'_{n,i}(t)|$$

$$\leqq \hat{C} \sum_i \inf_{t \in \mathscr{D}} |f'_{n,i}(t)| \leqq \hat{C} \sum_i \inf_{t \in I} |f'_{n,i}(t)|$$

$$= \hat{C} \sum_i \inf_{t \in I} f'_{n,i}(t) \leqq \hat{C} \inf_{t \in I} \sum_i f'_{n,i}(t)$$

$$= \hat{C} \inf_{t \in I} S_n(t) \leqq \hat{C}$$

where, as before, the last step follows from $\int_0^1 S_n(t)dt = 1$.

Thus the theorem is proved subject to (16); assume now that this restriction does not hold. By Theorem 3′ there must exist an N such that $T^* = T^N$ has slope everywhere > 1. Making the identification

$$f_i^* = f_{N,i}, \qquad i = 1, 2, \cdots p^N$$

we see that T^* conforms to (16) and is covered by our result. The iterated densities for T^* form a subsequence

$$S_0, S_N, S_{2N}, S_{3N}, \cdots$$

of the iterated densities for T; with this observation the proof is complete.

7. Some generalizations

It is not essential that the transformation T be increasing on each of the intervals of rank 1; let us allow it to be increasing on some, decreasing on others. (Such a T, of course, does not arise from a function ϕ as in section 2.) Our results go through with the addition of some absolute-value signs in appropriate places.

In a different direction, we generalize by relaxing the differentiability condition. Instead of requiring each $f_i \in C^2[0,1]$, it is sufficient to require that the first derivative be absolutely continuous, and that the second derivative be essentially bounded. We have shown ([2], pp. 26–28) that the boundedness of the second derivative is crucial. If s is fixed with $1 < s < 2$, then $T: x \to x + x^s \pmod 1$ cannot belong to $\mathcal{R}$ by Theorem 4; yet for this transformation we do have $\sum_{n=1}^{\infty} \Delta_n < \infty$. Thus theorem 2 would be false if T could be considered admissible.

Finally we turn our attention to transformations T mapping I onto itself *countably*-many-to-one. We make no attempt at a complete investigation, but simply indicate how our methods may be applied to particular examples.

Examination of the proofs shows the need for further hypotheses in the countable case. Thus, to push through the proof of Theorem 1 it will be sufficient to impose the additional requirement

$$\sup_i \sup_{t_1, t_2 \in I} \left| \frac{f_i''(t_1)}{f_i'(t_2)} \right| < \infty.$$

In Theorem 8, the inequality (12) now involves an infinite series, but can still be guaranteed for large N. However, the validity of (13) is not automatic; we require uniform (or at least dominated) convergence of the right hand side to justify the termwise differentiation. Note that the finiteness of d must now be specified. In Theorem 9 analogous new requirements appear. In Theorem 10 we verify directly the condition (17) (with sup replacing max) for some $\mathcal{D}(\varepsilon)$.

Let us consider now the transformation

$$T_\alpha : x \to \frac{1}{x^\alpha} \pmod 1, \qquad \alpha \geqq 1,$$

which for $\alpha = 1$ corresponds to the continued fraction. We have explicitly

$$f_i(t) = (i + t)^{-1/\alpha}$$

and it is readily seen that, for $\alpha > 1$, all the above requirements are met. For $\alpha = 1$ we work with the iterate T_1^2 and everything goes through. Thus we infer the analyticity on $[0, 1]$ of the invariant density $h_\alpha(t)$; for $\alpha = 1$ this is confirmed by the fact, known to Gauss, that

$$h_1(t) = \frac{1}{\log 2} \frac{1}{1+t}.$$

8. Acknowledgements

The present paper extends the results obtained originally in our dissertation [2]. This was written in June of 1974 at Oregon State University, under the guidance of our advisor, Dr. Arvid T. Lonseth. At this time we benefitted also from many helpful conversations with Dr. H. D. Brunk, of the Department of Statistics.

We are grateful to the department of Mathematical Sciences at the IBM Thomas J. Watson Research Center, for the granting of a postdoctoral fellowship for the year 1975–6, which has afforded us the opportunity to revise and improve this work. In particular we have profited from several instructive discussions with Drs. Roy L. Adler and Leopold Flatto.

References

1. Patrick Billingsley, *Ergodic Theory and Information*, John Wiley & Sons, New York, 1965.

2. Matthew Halfant, *Some Results in the Ergodic Theory of Generalized Expansions of Real Numbers*, Thesis, Oregon State University, 1974.

3. Sôichi Kakeya, *On a generalized scale of notations*, Japan J. Math. **1** (1924), 95–108.

4. R. O. Kuz'min *Sur un problème de Gauss*, Atti de Congresso Internazionale de Mathematici Bologna, Vol. VI, 1928, pp. 83-89.

5. Alfréd Rényi, *Representations for real numbers and their ergodic properties*, Acta. Math. Acad. Sci. Hungar. **8** (1957), 477–493.

6. C. Ryll-Nardzewski, *On the ergodic theorems II*, Studia Math. **12** (1951), 74–79.

7. E. C. Titchmarsh, *The Theory of Functions*, Oxford University Press, London, 1939.

IBM Thomas J. Watson Research Center
Yorktown Heights, New York 10598 USA